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Spectra of principal submatrices of nonnegative matrices ☆

Thomas J. Laffey, Helena Šmigoc *

School of Mathematical Sciences, University College Dublin, Dublin, Ireland

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Abstract

Let A be an $n \times n$ nonnegative matrix with the spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and let A_1 be an $m \times m$ principal submatrix of A with the spectrum $(\mu_1, \mu_2, \ldots, \mu_m)$. In this paper we present some cases where the realizability of $(\mu_1, \mu_2, \ldots, \mu_m, \nu_1, \nu_2, \ldots, \nu_s)$ implies the realizability of $(\lambda_1, \lambda_2, \ldots, \lambda_n, \nu_1, \nu_2, \ldots, \nu_s)$ and consider the question of whether this holds in general. In particular, we show that the list

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, -\mu_1, -\mu_2, \ldots, -\mu_m)$$

is realizable.

Keywords: Nonnegative matrices; Eigenvalues; Submatrices; Nonnegative inverse eigenvalue problem

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) is the problem of determining necessary and sufficient conditions for a list of complex numbers

$$\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

to be the spectrum of a nonnegative matrix. If a list σ is the spectrum of a nonnegative matrix A, we will say that σ is realizable and that the matrix A realizes σ.

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* Corresponding author.
E-mail addresses: thomas.laffey@ucd.ie (T.J. Laffey), Helena.Smigoc@fmf.uni-lj.si (H. Šmigoc).

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The problem was posed by Suleǐmanova [1] in 1949. It has been studied in its general form [2–9] as well as for the case when the list σ has real elements (RNIEP) [10–12], for the case when the matrices under consideration are symmetric (SNIEP) [13–16], for the case when the matrices have trace zero [17] and for other special cases.

Below are listed some necessary conditions on the list of complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix.

(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ.
(2) The list σ is closed under complex conjugation.
(3) $s_k = \sum_{i=1}^{n} \lambda_i^k \geqslant 0$.
(4) $s_k^m \leqslant n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$

The first condition listed above follows from the Perron–Frobenius theorem, which is a basic theorem in the theory of nonnegative matrices. The last condition was proved by Johnson [3] and independently by Loewy and London [2].

The necessary conditions we presented are sufficient only when the list σ has at most three elements. The solution to the NIEP has been found also for lists with four elements [2], while the problem for lists with five or more elements is still open.

Let A and B be nonnegative matrices such that $A - B$ is also nonnegative. Then the Perron eigenvalue of the matrix A is less than or equal to the Perron eigenvalue of the matrix B. In particular, that tells us that the Perron eigenvalue of any principal submatrix of a nonnegative matrix A is less than or equal to the Perron eigenvalue of A.

In this work we will present some results that demonstrate how we can use submatrices of nonnegative matrices to construct new realizable lists from known realizable lists. In Section 2 we will deal with the special case of 1×1 principal submatrices, i.e., diagonal elements, in Section 3 we will present some results for general submatrices and give some examples. We will finish the paper with two questions that are partially answered by the results in this paper, but they remain open in general.

2. Diagonal elements

In this section we discuss how to use the diagonal elements of the realizable matrices to find new realizable lists. The theorem below was proved for general nonnegative matrices in [8] and for symmetric nonnegative matrices in [18].

Theorem 1. *Let A be an $n \times n$ irreducible nonnegative (symmetric) matrix with the Perron eigenvalue λ_1, the spectrum $(\lambda_1, \ldots, \lambda_n)$ and the diagonal elements $(a_1, a_2, \ldots, a_n)$. Let B be an $m \times m$ nonnegative (symmetric) matrix with the Perron eigenvalue μ_1, the spectrum $(\mu_1, \mu_2, \ldots, \mu_m)$ and the diagonal elements $(b_1, b_2, \ldots, b_m)$.*

(1) *If $\mu_1 \leqslant a_n$, then there exists an $(n + m - 1) \times (n + m - 1)$ nonnegative (symmetric) matrix C with the spectrum*

$$(\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m) \tag{1}$$

and the diagonal elements $(a_1, a_2, \ldots, a_{n-1}, b_1, \ldots, b_m)$.

(2) *If $a_n \leqslant \mu_1$, then there exists an $(n + m - 1) \times (n + m - 1)$ nonnegative (symmetric) matrix C with the spectrum*

$$(\lambda_1 + \mu_1 - a_n, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m) \tag{2}$$

and the diagonal elements $(a_1, a_2, \ldots, a_{n-1}, b_1, \ldots, b_m)$.

Notice that the diagonal elements of the matrix A in the theorem are not ordered. That means that we can choose any diagonal element of the matrix A to play the role of a_n in the result.

Theorem 2 (Šmigoc [8]). *Let A be a nonnegative $n \times n$ matrix with spectrum σ_0 and maximal diagonal element c_0. Let B_i, $i = 1, \ldots, k$, be nonnegative $m_i \times m_i$ matrix with maximal diagonal element c_i, Perron eigenvalue ρ_i and spectrum (ρ_i, σ_i), where σ_i is a list of complex numbers. If*

$$c_0 + c_1 + \cdots + c_{i-1} \geqslant \rho_1 + \rho_2 + \cdots + \rho_i \tag{3}$$

for $i = 1, \ldots, k$, then there exists a nonnegative matrix C with spectrum $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ and maximal diagonal element greater than or equal to

$$c_0 + c_1 + \cdots + c_k - \rho_1 - \rho_2 - \cdots - \rho_k. \tag{4}$$

Suppose $\sigma = (\rho, p_1 + \mathrm{i}q_1, p_1 - \mathrm{i}q_1, \ldots, p_r + \mathrm{i}q_r, p_r - \mathrm{i}q_r)$, where $\rho > 0$, $p_i \geqslant 0, q_i \geqslant 0, i = 1, \ldots, r$. We want to find a good estimate for ρ that makes the list σ realizable.

We can realize $(p_i + q_i\sqrt{3}, p_i + \mathrm{i}q_i, p_i - \mathrm{i}q_i)$ by a matrix A_i, say, and A_i has entries $\frac{3p_i + q_i\sqrt{3}}{3}$ along the diagonal. The solution to the NIEP for $n = 3$ tells us that $p_i + q_i\sqrt{3}$ is the smallest Perron eigenvalue ρ_i that makes the list $(\rho_i, p_i + \mathrm{i}q_i, p_i - \mathrm{i}q_i)$ realizable. Theorem 2 gives us the following estimate for the Perron eigenvalue.

Theorem 3. *Let p_i and q_i, $i = 1, \ldots, k$, be nonnegative real numbers that satisfy the following inequality*:

$$\rho \geqslant \max\left\{p_1 + \sqrt{3}q_1, p_2 + \sqrt{3}q_2 + \frac{2\sqrt{3}}{3}q_1, \ldots, p_k + \sqrt{3}q_k + \frac{2\sqrt{3}}{3}(p_1 + \cdots + p_{k-1})\right\}. \tag{5}$$

Then the list $(\rho, p_1 + \mathrm{i}q_1, p_1 - \mathrm{i}q_1, \ldots, p_k - \mathrm{i}q_k, p_k + \mathrm{i}q_k)$ is realizable.

Proof. Let $(p_i + \sqrt{3}q_i, p_i + \mathrm{i}q_i, p_i - \mathrm{i}q_i)$ be the spectrum of a nonnegative matrix B_i. Then B_i has all its diagonal elements equal to $p_i + \frac{\sqrt{3}}{3}q_i$. We insert $\rho_i = p_i + \sqrt{3}q_i$ and $c_i = p_i + \frac{\sqrt{3}}{3}q_i$ in Theorem 2 to finish the proof. □

Careful analysis of the diagonal elements of the realizing matrices leads to a better estimation for the Perron eigenvalue. Let's assume that the elements in the list σ are ordered in the following way:

$$p_1 + \frac{\sqrt{3}}{3}q_1 \geqslant p_2 + \frac{\sqrt{3}}{3}q_2 \geqslant \cdots \geqslant p_k + \frac{\sqrt{3}}{3}q_k. \tag{6}$$

We define:

$$\rho_1 = p_1 + \sqrt{3}q_1 \tag{7}$$

and inductively

$$\rho_i = \rho_{i-1} + \max\left\{0, p_i + \sqrt{3}q_i - p_{\left\lfloor \frac{i+1}{3} \right\rfloor} - \frac{\sqrt{3}}{3}q_{\left\lfloor \frac{i+1}{3} \right\rfloor}\right\} \tag{8}$$

for $i = 2, \ldots, k$.

We define the lists:

$$\mathscr{D}'_i = \left(p_i + \frac{\sqrt{3}}{3}q_i,\, p_i + \frac{\sqrt{3}}{3}q_i,\, p_i + \frac{\sqrt{3}}{3}q_i\right) \tag{9}$$

for $i = 1, \ldots, k$. Let $\mathscr{D}_1 = \mathscr{D}'_1$. For $i = 2, \ldots, k$ we define the lists $\mathscr{D}_i$ inductively. The list $\mathscr{D}_i$ is obtained from the list $\mathscr{D}_{i-1}$ in the following way. Remove one instance of $p_{\left\lfloor \frac{i+1}{3} \right\rfloor} + \frac{\sqrt{3}}{3}q_{\left\lfloor \frac{i+1}{3} \right\rfloor}$ from the list $\mathscr{D}_{i-1}$ and add the list $\mathscr{D}'_i$ to it.

Theorem 4. *Let p_j, q_j, $j = 1, \ldots, k$, be nonnegative real numbers ordered as in* (6) *and let ρ_k be inductively defined by* (7) *and* (8). *Then the list*

$$(\rho_k, p_1 + \mathrm{i}q_1, p_1 - \mathrm{i}q_1, \ldots, p_k - \mathrm{i}q_k, p_k + \mathrm{i}q_k) \tag{10}$$

is the spectrum of a nonnegative matrix with the diagonal elements $\mathscr{D}_k$.

Proof. We will prove the proposition by induction on k. For $k = 1$ the proposition follows from the solution to the NIEP for lists with three elements. It is easy to check that $p_{\left\lfloor \frac{i+1}{3} \right\rfloor} + \frac{\sqrt{3}}{3}q_{\left\lfloor \frac{i+1}{3} \right\rfloor}$ is an element of the list $\mathscr{D}_{i-1}$ for $i = 2, \ldots, k$.

Now we assume that the proposition holds for $k - 1$. In this case the list

$$(\rho_{k-1}, p_1 + \mathrm{i}q_1, p_1 - \mathrm{i}q_1, \ldots, p_{k-1} - \mathrm{i}q_{k-1}, p_{k-1} + \mathrm{i}q_{k-1}) \tag{11}$$

is the spectrum of a nonnegative matrix with a diagonal element $p_{\left\lfloor \frac{k+1}{3} \right\rfloor} + \frac{\sqrt{3}}{3}q_{\left\lfloor \frac{k+1}{3} \right\rfloor}$. We join this list with the list $(p_k + \sqrt{3}q_k, p_k + \mathrm{i}q_k, p_k - \mathrm{i}q_k)$ using Theorem 1, to finish the proof. □

The example below illustrates that Theorem 4 gives us a better estimate for the Perron eigenvalue than Theorem 3.

Example 5. Let

$$\sigma = (\rho, 3 + \mathrm{i}\sqrt{3}, 3 - \mathrm{i}\sqrt{3}, 1 + \mathrm{i}2\sqrt{3}, 1 - \mathrm{i}2\sqrt{3}, 1 + \mathrm{i}\sqrt{3}, 1 - \mathrm{i}\sqrt{3}). \tag{12}$$

We want to find a good estimate for the Perron eigenvalue ρ that will make the list σ realizable.

Theorem 3 gives us

$$\rho = \max\{6, 9, 10\} = 10. \tag{13}$$

The numbers in the list are already ordered as we have assumed in Theorem 4. Now we compute ρ_i as described above: $\rho_1 = 6$, $\rho_2 = 9$ and $\rho_3 = 9$. We conclude that the list σ is realizable for $\rho = 9$.

3. Principal submatrices

We begin this section with a simple observation on how to construct a nonnegative matrix from a set of complex matrices. Even though the conditions on the matrices B_k in the proposition below are very restrictive, we will later show how the proposition can be used to obtain some interesting results.

Proposition 6. *Let $B_k = (b_{ij}(k))$ be $n \times n$ complex matrices with spectrum σ_k for $k = 1, \dots, m$. Assume that there exists an invertible matrix T such that for every pair (i, j) the matrix*

$$C_{ij} = T \begin{pmatrix} b_{ij}(1) & 0 & \cdots & 0 \\ 0 & b_{ij}(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{ij}(m) \end{pmatrix} T^{-1} \tag{14}$$

is nonnegative with the Perron eigenvalue $b_{ij}(1)$. Then the list $(\sigma_1, \sigma_2, \dots, \sigma_m)$ is realizable.

Proof. Observe that the matrix

$$(T \otimes I_n) \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix} (T^{-1} \otimes I_n)$$

is nonnegative. □

Remark 7. If the matrices B_k, $k = 1, \dots, m$, are symmetric and the matrix T is orthogonal then the list $(\sigma_1, \sigma_2, \dots, \sigma_m)$ is realizable by a symmetric nonnegative matrix.

Proposition 6 used for two real matrices gives us the following corollary.

Corollary 8. *Let $B_1 = (b_{ij}(1))$ and $B_2 = (b_{ij}(2))$ be $n \times n$ real matrices, such that $B_1 \geqslant |B_2|$, i.e., $b_{ij}(1) \geqslant |b_{ij}(2)|$ for all i, j. Then the list $\sigma = (\sigma_1, \sigma_2)$ is realizable.*

Proof. Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then the matrix

$$T \begin{pmatrix} b_{ij}(1) & 0 \\ 0 & b_{ij}(2) \end{pmatrix} T^{\mathrm{T}} = \frac{1}{2} \begin{pmatrix} b_{ij}(1) + b_{ij}(2) & b_{ij}(1) - b_{ij}(2) \\ b_{ij}(1) - b_{ij}(2) & b_{ij}(1) + b_{ij}(2) \end{pmatrix}$$

is nonnegative for every pair (i, j). Proposition 6 tells us that the list (σ_1, σ_2) is realizable. If matrices B_1 and B_2 are symmetric, then the list (σ_1, σ_2) is symmetrically realizable. □

Corollary below is obtained if we take T in Proposition 6 to be a Soules matrix. Soules matrices were first introduced in [13].

Corollary 9. *Let matrices B_i, $i = 1, \dots, m$, satisfy the following conditions:*

(1) $B_1 \geqslant B_2 \geqslant B_3 \geqslant \cdots \geqslant B_m$,
(2) $\frac{1}{m} B_1 + \frac{1}{m(m-1)} B_2 + \frac{1}{(m-1)(m-2)} B_3 + \cdots + \frac{1}{2} B_m \geqslant 0$,

and let σ_i be the spectrum of B_i for $i = 1, 2, \dots, m$. Then the list $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ is realizable. If the matrices B_i are symmetric, then the list σ is symmetrically realizable.

Proof. Let real numbers b_i, $i = 1, \ldots, m$, satisfy the following conditions:

(1) $b_1 \geqslant b_2 \geqslant b_3 \geqslant \cdots \geqslant b_m$,
(2) $\frac{1}{m}b_1 + \frac{1}{m(m-1)}b_2 + \frac{1}{(m-1)(m-2)}b_3 + \cdots + \frac{1}{2}b_m \geqslant 0$.

In [13] an orthogonal matrix R was constructed, such that the matrix

$$R\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_m \end{pmatrix} R^{\mathrm{T}} \geqslant 0$$

is nonnegative for all lists of real numbers b_i that satisfy the above conditions. Hence matrices B_i satisfy the conditions listed in Proposition 6 and this completes the proof. $\square$

Remark 10. We can obtain similar results by looking at matrices B_i, $i = 1, 2, \ldots, m$, that satisfy the conditions in Proposition 6 for any given Soules matrix R.

The theorem below introduces a way to use the spectra of principal matrices of nonnegative matrices, to construct new realizable lists.

Theorem 11. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be an $(n_1 + n) \times (n_1 + n)$ nonnegative matrix with spectrum σ, where A_{11} is an $n_1 \times n_1$ matrix. Let matrices $B_1 = A_{22}, B_2, \ldots, B_m$ satisfy the conditions in Proposition 6 for some invertible matrix T. Let σ_j be the spectrum of B_j for $j = 1, \ldots, m$. Then the list $(\sigma, \sigma_2, \ldots, \sigma_m)$ is realizable.

Proof. Let

$$\widehat{T} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & T \otimes I_n \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_2 & 0 & \cdots & 0 \\ 0 & B_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{pmatrix}.$$

Since the matrices C_{ij} in (14) are nonnegative and have the Perron eigenvalue $b_{ij}(1)$, the matrix T has nonnegative first column s_1 and the matrix T^{-1} has nonnegative first row t_1^{T}. Let

$$C = (T \otimes I_n)\begin{pmatrix} A_{22} & 0 \\ 0 & B \end{pmatrix}(T^{-1} \otimes I_n).$$

As in the proof of Proposition 6, the matrix C is nonnegative.

Now we have

$$\widehat{T}\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & B \end{pmatrix}\widehat{T}^{-1} = \begin{pmatrix} A_{11} & A_{12}(t_1^{\mathrm{T}} \otimes I_n) \\ (s_1 \otimes I_n)A_{21} & C \end{pmatrix}.$$

Since the above matrix is nonnegative, we have finished the proof. $\square$

Remark 12. If the matrices A and $B_i, i = 1, \dots, m$, in Theorem 11 are symmetric and the matrix T is orthogonal we obtain a symmetric realization.

Now we use Corollaries 8 and 9 to obtain the following two results.

Corollary 13. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be an $(n_1 + n) \times (n_1 + n)$ *nonnegative matrix with spectrum* σ. *Let* B_2 *be an* $n \times n$ *matrix with spectrum* σ_2, *such that* $A_{22} \geqslant |B_2|$. *Then the list* (σ, σ_2) *is realizable. In particular, let* $\sigma(A_{22})$ *denote the spectrum of the matrix* A_{22}, *then the list* $(\sigma, -\sigma(A_{22}))$ *is realizable.*

Proof. Corollary 8 and Theorem 11 give us the result. $\square$

Corollary 14. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be an $(n_1 + n) \times (n_1 + n)$ *nonnegative matrix with spectrum* σ. *Let matrices* B_i *with spectrum* $\sigma_i, i = 2, \dots, m$, *satisfy the following conditions*:

(1) $A_{22} \geqslant B_2 \geqslant B_3 \geqslant \cdots \geqslant B_m$,
(2) $\frac{1}{m} A_{22} + \frac{1}{m(m-1)} B_2 + \frac{1}{(m-1)(m-2)} B_3 + \cdots + \frac{1}{2} B_m \geqslant 0$.

Then the list $(\sigma, \sigma_2, \dots, \sigma_m)$ *is realizable. If the matrices* A *and* B_i, $i = 2, \dots, m$, *are symmetric, then the list* σ *is symmetrically realizable.*

Proof. The result follows from Corollary 9 and Theorem 11. $\square$

Let $\alpha \subseteq N = \{1, 2, \dots, n\}$ and A be an $n \times n$ matrix, in the examples below we will let $A[\alpha]$ denote the principal submatrix of A lying in the rows and columns indexed by α.

Perron–Frobenius theorem tells us, that if a nonnegative matrix has repeated Perron eigenvalue, then it is reducible. An open question is, how close can an eigenvalue of a nonnegative matrix be to a Perron eigenvalue, in order that we can have an irreducible realization. One very well studied example is $\sigma = (3 + t, 3, -2, -2, -2)$. We know, that for $t = 0$, σ is not realizable, however the smallest t for which the list is realizable is not known. Some estimates are given in [19]. In [18] the following example was discussed.

Example 15. It follows from results in [20] that the smallest t for which $(3 + t, 3 - t, -2, -2, -2)$ is the spectrum of a symmetric nonnegative matrix is $t = 1$. Consider the question for which t is $(3 + t, 3 - t, 0, -2, -2, -2)$ the spectrum of a symmetric nonnegative matrix. Matrix

$$\begin{pmatrix} 2 & 0 & 2\sqrt{\frac{2}{3}} & 0 \\ 0 & 2 & 0 & 2\sqrt{\frac{2}{3}} \\ 2\sqrt{\frac{2}{3}} & 0 & 0 & \frac{4}{3} \\ 0 & 2\sqrt{\frac{2}{3}} & \frac{4}{3} & 0 \end{pmatrix}$$

has the spectrum $(\frac{10}{3}, \frac{8}{3}, 0, -2)$.

Since $A[1, 2]$ has spectrum $(2, 2)$, Corollary 13 tells us that the list $(\frac{10}{3}, \frac{8}{3}, 0, -2, -2, -2)$ is symmetrically realizable and that shows us that the smallest t that realizes $(3 + t, 3 - t, 0, -2, -2, -2)$ is smaller than or equal to $1/3$.

Similarly, since $A[1, 2, 3]$ has spectrum $(2, \frac{1}{3}(3 - \sqrt{33}), \frac{1}{3}(3 + \sqrt{33}))$, Corollary 13 tells us that the list

$$\left(\frac{10}{3}, \frac{8}{3}, 0, -2, -2, -\frac{1}{3}(3 - \sqrt{33}), -\frac{1}{3}(3 + \sqrt{33})\right)$$

is symmetrically realizable.

Example 16. Let

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ c_n & c_{n-1} & \cdots & c_2 & c_1 \end{pmatrix} \tag{15}$$

be a nonnegative companion matrix with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $m \leqslant n$ and let $(\mu_1, \mu_2, \ldots, \mu_m)$ be the roots of the polynomial

$$f(x) = x^m - d_1x^{m-1} - d_2x^{m-2} - \cdots - d_m,$$

where d_i are real numbers such that $c_i \geqslant |d_i|$. Then Corollary 13 tells us that the list

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_m)$$

is realizable.

We conclude this work with two questions.

Problem 1. Let A be an irreducible nonnegative matrix with spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and let $A(i_1, i_2, \ldots, i_k)$ be a $k \times k$ principal submatrix of A with spectrum $(\mu_1, \mu_2, \ldots, \mu_k)$. Let the list $(\mu_1, \mu_2, \ldots, \mu_k, \nu_1, \nu_2, \ldots, \nu_s)$ be realizable. Is then the list $(\lambda_1, \lambda_2, \ldots, \lambda_n, \nu_1, \nu_2, \ldots, \nu_s)$ realizable?

Problem 2. Let A be an irreducible nonnegative symmetric matrix with spectrum

$$(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and let $A(i_1, i_2, \ldots, i_k)$ be a $k \times k$ principal submatrix of A with spectrum $(\mu_1, \mu_2, \ldots, \mu_k)$. Let the list $(\mu_1, \mu_2, \ldots, \mu_k, \nu_1, \nu_2, \ldots, \nu_s)$ be realizable. Is then the list $(\lambda_1, \lambda_2, \ldots, \lambda_n, \nu_1, \nu_2, \ldots, \nu_s)$ symmetrically realizable?

Clearly both questions have a positive answer when $k = n$ and in the Section 2 it is shown that the questions have a positive answer when $k = 1$. In Section 3 we have presented some special cases for an arbitrary k, but in general both questions remain open.

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